Split Generalized Variational Inequality and Mixed Equilibrium Problem

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Abstract: In this article, a neo-iterated scheme is constructed to settle split generalized variational inequality and mixed equilibrium problem in two different Hilbert spaces. Under several mild conditions, the sequence produced of the proposed iterated algorithm converges strongly to solution of split generalized variational inequality and mixed equilibrium problem is proved. As application, we shall apply our result to reserach the split variational inequality problem and convex minimization problem. The results received in this paper enhance and generalize a number of recent relevant results.

1. Introduction

Suppose $\mathcal{W}$ is a real Hilbert space, Y is a closed convex non-empty subset of $\mathcal{W}$. the problem of generalized variational inequality (GVI) is to seek $c \in Y$ satisfies

$$\langle Kc, t(d) - t(c) \rangle \geq 0, \forall t(d) \in Y, \tag{1}$$

Where $K : \mathcal{W} \to \mathcal{W}$ be a non-linear opreator, $t : \mathcal{W} \to \mathcal{W}$ be a continuous operator. $GVI(K, t, Y)$ represents the solution set of (1).

If $t = I$, problem (1) simplified the variational inequality problem, which is considred to seek $c \in Y$ satisfies

$$\langle Kc, d - c \rangle \geq 0, \forall d \in Y, \tag{2}$$

$VI(K, Y)$ represents the solution set of (2).

Stampacchia [1] and Fichera [2] introduced Variational inequality theory, which furnishes the unified, natural, descent and valid structure for a ordinary treatment of a broad category of extraneous linear and non-linear problem proceed from transportations, elasticity, economics, engineering sciences, optimization and control theory, see for instance [3-8].

The mixed equilibrium problem (MEP) is to seek $c \in Y$ satisfies

$$G(c, d) + \varphi(d) - \varphi(c) \geq 0, \ \forall d \in Y, \tag{3}$$

Where $G : Y \times Y \to R$ is a nonlinear bifunction, $\varphi : Y \to R \cup \{+\infty\}$ is a function with $C \cap dom\varphi \neq \varnothing$. $MEP(G, \varphi)$ represents the solution of (3).

If $\varphi = 0$, the mixed equilibrium problem of (3) down to the equilibrium problem, which is to seek $c \in Y$ satisfies

$$G(c, d) \geq 0, \ \forall d \in Y, \tag{4}$$

$EP(G)$ represents the solution set of (4).

The mixed equilibrium problem covers serval significant problems arising in science optimization, economics, physics, engineering, structural analysis, transportation and network, It has been demonstrated that mathematical programming problems can be thinked of as a prticular accomplishment of the abstract equilibrium problems (e.g. [9,10]).

Recently, split feasibility problem (SFP), which was first presented by Censor and Elfving [11], has been widely concerned due to its utilizations in diverse fields, for instance, computer tomograph, image restoration and radiation therapy treatment planning [12-14]. SFP is considered the problem of seeking a point c satisfies

$c \in Y$ and $Ac \in D$,

Where Y and D be closed convex nonempty subset of real Hilbert spaces $\mathcal{W}_1$ and $\mathcal{W}_2$,respectively. $A: \mathcal{W}_1 \to \mathcal{W}_2$ is a linear operator with bounded .

After split feasibility problem appeared, many authors used its idea to study more generalized split feasibility problem, such as split equilibrium problem, split common fixed point problem, split variational inequality problem and so on, for details see [15-18].

In this article, we research following split generalized variational inequality and mixed equilibrium problem in two distrint Hilbert spaces: seek a point $c \in Y$ satisfies

$$c \in GVI(K,t,Y) \quad \text{and} \quad Ac \in MEP(G,\varphi) \tag{5}$$

We construct an iterated algorithm and obtain strong convergence theorem. The main results recived in this paper enhance and generalize a number of relevant result.

2. Preliminaries

The inner product denoted by $\langle \cdot,\cdot \rangle$, the norm denoted by $\|\cdot\|$.

We call a mapping $T: Y \to \mathcal{W}$

(a) monotone, if

$\langle Te - Td, e-d \rangle \geq 0, \quad \forall e,d \in Y;$

(b) strongly monotone, if $\gamma > 0$ satisfies

$\langle Te - Td, e-d \rangle \geq \gamma \|e-d\|^2, \ \ \forall e,d \in Y;$

(c) ϖ -inverse strongly nonotone, if there is $\varpi > 0$ such that

$\langle Te - Td, e-d \rangle \geq \varpi \|Te - Td\|^2, \ \ \forall e,d \in Y;$

(d) ϖ -inverse strongly t-monotone, if thereis $\varpi > 0$ and a nonlinear operator t from Y into itself such that

$\langle t(e) - t(d, Te - Td \rangle \geq \varpi \|Te - Td\|^2, \ \ \forall e,d \in Y;$

(e) L-Lipschitz continuous, if there exists a constant L > 0 such that

$\|Te - Td\| \leq L\|e-d\|, \ \forall e,d \in Y;$

(f) firmly nonexpansive, if

$\langle Te - Td, e-d \rangle \geq \|Te - Td\|^2, \ \forall e,d \in C;$

A multi-valued mapping $U: \mathcal{W}_1 \to 2^{\mathcal{W}_1}$ is monotone, if $\forall e,d \in \mathcal{W}_1, \ \theta \in Ue$ and $v \in Ud$ satisfy $\langle e-d, \theta - v \rangle \geq 0.$ A monotone mapping $U: \mathcal{W}_1 \to 2^{\mathcal{W}_1}$ is said to be maximal if the Graph(U) cannot be properly included in the graph of any other monotone mapping.

A monotone mapping U is called maximal iff $(e,\theta) \in \mathcal{W}_1 \times \mathcal{W}_1, \ \langle e-d, \theta - v \rangle \geq 0,$ for each $(d,v) \in$ Graph(U) implies that $\theta \in Ue$ and a mapping U is maximal g- monotone when and only when for $(e,u)\in \mathcal{W}_1\times\mathcal{W}_1$, $\langle g(e)-g(d),u-v\rangle\geq 0,$ for each $(d,v)\in$ Graph(U) implies that $u\in Ue$.

We suppose bifunction $G:Y\times Y\to R$, φ and the set Y satisfy the following conditions to solve mixed equilibrium problem (3):

(B1) $G(e,e)=0, \forall e\in Y;$

(B2) G is monontone, i.e. $G(e,q)+G(q,e)\leq 0, \forall e,q\in Y;$

(B3) For all $e,q,\alpha\in Y$, $\lim_{t\downarrow 0} G(t\alpha+(1-t)e,q)\leq G(e,q);$

(B4) For every $e\in Y$, the function $q\mapsto G(e,q)$ is convex and lower semi-continuous.

(C1) For every $e\in Y$, $\xi>0$, there is a subset with bounded $Y_e\subseteq Y$ and $q_e\in Y\cap dom\varphi$ satisfies $G(e,q_e)+\varphi(q_e)+\frac{1}{\xi}\langle q_e-b,b-e\rangle\leq\varphi(b), \ \forall b\in Y\setminus Y$

(C2) Y is a bounded set.

Lemma 2.1. ([19]) Suppose $G:Y\times Y\to R$ is a bifunction and satisfies the conditions (B1)-(B4), $\varphi:Y\to R\cup\{+\infty\}$ is a convex and lower semicontinuous proper function satisfying $Y\cap dom\varphi\neq\varnothing$. For $\xi>0$, $e\in Y$, define a operator $S:Y\to \mathcal{W}$ as below:

$$S(e)=\left\{\alpha\in Y: G(e,q)+\varphi(q)+\frac{1}{\xi}\langle q-\alpha,\alpha-e\rangle\geq\varphi(\alpha),\ \forall q\in Y\right\} \tag{6}$$

For every $e\in\mathcal{W}$. Suppose both (C1) and (C2) are true. Then the listed below conclusions hold:

(a) For every $e\in\mathcal{W}$, $S(e)\neq\varnothing;$

(b) S is firmly nonexpansive;

(c) S is single-valued;

(d) $F(S)=MEP(G,\varphi);$

(e) $MEP(G,\varphi)$ is convex and closed.

Lemma 2.2. ([20]) Suppose $T:\mathcal{W}\to\mathcal{W}$ is a nonexpansive mapping, then T has the listed below properties:

(1) $\forall(e,d)\in\mathcal{W}\times\mathcal{W}$, we have

$$\langle(e-Te)-(d-Td),Td-Te\rangle\leq\frac{1}{2}\|(Te-e)-(Td-d)\|^2, \tag{7}$$

$$\langle(Te-e)-(Td-d),d-e\rangle\leq-\frac{1}{2}\|(Te-e)-(Td-d)\|^2, \tag{8}$$

(2) $\forall(e,d)\in\mathcal{W}\times Fix(T)$, we have

$$\langle(e-Te),d-Te\rangle\leq\frac{1}{2}\|Te-e\|^2. \tag{9}$$

Lemma 2.3. ([21]) There are bounded sequences $\{e_n\}$, $\{d_n\}$ in a Banach space E, let sequence $\{\zeta_n\}\in[0,1]$ satisfies $0<\liminf_{n\to\infty}\zeta_n\leq\limsup_{n\to\infty}\zeta_n<1$. Assume $e_{n+1}=(1-\zeta_n)d_n+\zeta_n e_n$ for every $n\geq 0$ and $\limsup_{n\to\infty}(\|d_{n+1}-d_n\|-\|e_{n+1}-e_n\|)\leq 0.$ Then $\lim_{n\to\infty}\|d_n-e_n\|=0.$

Lemma 2.4. ([22]) Suppose sequence $\{\beta_n\}$ is nonnegative real numbers and satisfies

$\beta_{n+1} \le (1-\varepsilon_n)\beta_n + \varepsilon_n \gamma_n$, here sequence $\{\varepsilon_n\} \in (0,1)$ and $\sum_{n=1}^{\infty} \varepsilon_n = \infty$, $\{\gamma_n\}$ is a sequence with $\limsup_{n\to\infty} \gamma_n \le 0$ (or $\sum_{n=1}^{\infty} |\varepsilon_n \gamma_n| < \infty$). Then $\lim_{n\to\infty} \beta_n = 0$.

Lemma 2.5. ([23]) Let $B_r(0):\{e \in E : \|e\| \le r\}$ be a closed ball with center 0 and radius $r>0$ in uniformly convex Banach space E. For given arbitrarily sequence $\{e_1, e_2, \cdots, e_n, \cdots\} \subset B_r(0)$ and given arbitrarily number sequence $\{\mu_1, \mu_2, \cdots, \mu_n, \cdots\}$ such that $\mu_i \ge 0, \sum_{i=1}^{\infty} \mu_i = 1,$ then there is a convex and continuous strictly increasing function $h:[0,2r) \to [0,\infty)$ with $h(0)=0$ such that for any $i, j \in N, i<j$ the below inequalty is true:

$$\left\| \sum_{n=1}^{\infty} \mu_n e_n \right\|^2 \le \sum_{n=1}^{\infty} \mu_n \|e_n\|^2 - \mu_i \mu_j h\left(\|e_i - e_j\|\right). \tag{10}$$

Lemma 2.6.([24]) Suppose $S: Y \to Y$ be a nonexpansive mapping, then $I-S$ is demi-closed at zero, that is to say, for every sequence $\{e_n\}$ in Y. if $\{e_n\}$ converges weakly to $p \in Y$ and $\{(I-S)e_n\}$ converges strongly to 0, then $(I-S)p=0$.

3. Main Results

In this part, we suppose the listed below conditions are met:

(1) Assume that Y and D be two closed convex and non-empty subsets of real Hilbert spaces $\mathcal{W}_1$ and $\mathcal{W}_2$, severally;

(2) $K: Y \to Y$ is an ϖ-inverse strongly t-monotone mapping; $t: Y \to Y$ is a σ-strongly monotone and L-Lipschitz continuous mapping with $Y = R(t)$ (the range of t); $G: D \times D \to R$ is a bifunction; $A: \mathcal{W}_1 \to \mathcal{W}_2$ is a linear and bounded operator, the adjoint operator of A is A^*.

(3) $\lambda \in (0, 2\varpi), \sigma > L, \gamma \in (0, \frac{1}{M}),$ where M is the spectral radius of AA^*.

Now, we present the main result as below:

Theorem 3.1. Let $\mathcal{W}_1, \mathcal{W}_2, Y, D, \lambda, \gamma, \alpha, L, M, G, A, t$ and K be the same as above. Assume that S is defined as in (6), P_Y is a metric projection of $\mathcal{W}_1$ onto Y. Let $\{e_n\}$, $\{b_n\}$ and $\{d_n\}$ be the sequences defined by

$$\begin{cases} e_1 \in Y, u \in Y, \\ t(d_n) = P_Y(t(e_n) - \lambda K e_n), \\ b_n = P_Y(d_n + \gamma A^*(S-I)(Ad_n)), \\ e_{n+1} = g_n u + l_n e_n + y_n b_n. \end{cases} \tag{11}$$

where $\{g_n\}$, $\{l_n\}$, $\{y_n\}$ satisfy the listed below conditions:

(i) $g_n + l_n + y_n = 1$;

(ii) $\lim_{n\to\infty} g_n = 0$;

(iii) $\sum_{n=0}^{\infty} g_n = \infty$;

(iv) $0 < \liminf_{n\to\infty} l_n \le \limsup_{n\to\infty} l_n < 1$.

If $\Omega \ne \varnothing$, where $\Omega = \{ p : p \in GVI(K,t,Y), Ap \in MEP(G,\varphi) \}$, then $\{e_n\}$ converges strongly to $\hat{k} = P_\Omega u.$

Proof. Let us break the proof down into several steps.

Step 1. We will first prove that $\tilde{e} \in \mathcal{H}_1$ is a solution of $GVI(K,t,Y) \Leftrightarrow$

$$t(\tilde{e}) = P_Y(t(\tilde{e}) - \lambda K\tilde{e}),\ \forall \lambda > 0, \tag{12}$$

By using the characteristic inequality of the projection, for any $d \in Y,$ we have

$$\begin{aligned} \tilde{e} \in GVI(K,t,Y) &\Leftrightarrow \left\langle K\tilde{e}, t(d) - t(\tilde{e}) \right\rangle \ge 0 \\ &\Leftrightarrow \left\langle \lambda K\tilde{e}, t(d) - t(\tilde{e}) \right\rangle \ge 0 \\ &\Leftrightarrow \left\langle t(\tilde{e}) - \lambda K\tilde{e} - t(\tilde{e}), t(d) - t(\tilde{e}) \right\rangle \le 0 \\ &\Leftrightarrow t(\tilde{e}) = P_Y(t(\tilde{e}) - \lambda K\tilde{e}). \end{aligned}$$

Step 2. Showing $\|t(e) - \lambda Ke - (t(d) - \lambda Kd)\|^2 \le \|t(e) - t(d)\|^2 + \lambda(\lambda - 2\varpi)\|Ke - Kd\|^2$.

In fact

$$\begin{aligned} \|t(e) - \lambda Ke - (t(d) - \lambda Kd)\|^2 &= \|t(e) - t(d)\|^2 - 2\lambda \left\langle Ke - Kd, t(e) - t(d) \right\rangle + \lambda^2 \|Ke - Kd\|^2 \\ &\le \|t(e) - t(d)\|^2 - 2\varpi\lambda \|Ke - Kd\|^2 + \lambda^2 \|Ke - Kd\|^2 \\ &= \|t(e) - t(d)\|^2 + \lambda(\lambda - 2\varpi)\|Ke - Kd\|^2. \end{aligned} \tag{13}$$

Next, we prove $\{e_n\}$ converges strongly to $\hat{b} = P_\Omega u.$

Step 3. We prove that $\|b_n - c\| \le \|e_n - c\|.$

Let $c \in \Omega,$ hence $t(c) = P_Y(t(c) - \lambda Kc)$ by (12), further, it follows from Lemma2.1, $SAc = Ac.$

By (13) and condition (3) we have

$$\begin{aligned} \|t(d_n) - t(c)\| &= \|P_Y(t(e_n) - \lambda K(e_n)) - P_Y(t(c) - \lambda Kc)\| \\ &\le \|t(e_n) - \lambda K(e_n) - (t(c) - \lambda Kc)\| \\ &\le \|t(e_n) - t(c)\| \\ &\le L\|e_n - c\|, \end{aligned} \tag{14}$$

Since t is σ-strongly monotone mapping, we get

$$\begin{aligned} \sigma\|d_n - c\|^2 &\le \left\langle d_n - c, t(d_n) - t(c) \right\rangle \\ &\le \|d_n - c\|\|t(d_n) - t(c)\|, \end{aligned}$$

So, by (14) and condition (3) we know

$$\begin{aligned} \|d_n - c\| &\le \frac{1}{\sigma}\|t(d_n) - t(c)\| \\ &\le \frac{L}{\sigma}\|e_n - c\| \\ &\le \|e_n - c\|. \end{aligned} \tag{15}$$

It follows (11) that

$$\|b_n - c\|^2 \le \|d_n + \gamma A^*(S-I)(Ad_n) - c\|^2$$
$$= \|d_n - c\|^2 + \gamma^2 \|A^*(S-I)(Ad_n)\|^2 + 2\gamma \left\langle d_n - c, A^*(S-I)(Ad_n) \right\rangle, \tag{16}$$

On the other hand, we have

$$\|A^*(S-I)(Ad_n)\|^2 = \left\langle (S-I)(Ad_n), AA^*(S-I)(Ad_n) \right\rangle$$
$$\le M \|(S-I)(Ad_n)\|^2, \tag{17}$$

And from Lemma 2.2, we have

$$\left\langle d_n - c, A^*(S-I)(Ad_n) \right\rangle = \left\langle A(d_n - c), (S-I)(Ad_n) \right\rangle$$
$$= \left\langle A(d_n - c) + (S-I)(Ad_n) - (S-I)(Ad_n), (S-I)(Ad_n) \right\rangle$$
$$= \left\langle SAd_n - Ac, (S-I)(Ad_n) \right\rangle - \|(S-I)(Ad_n)\|^2$$
$$\le \frac{1}{2}\|(S-I)(Ad_n)\|^2 - \|(S-I)(Ad_n)\|^2 = -\frac{1}{2}\|(S-I)(Ad_n)\|^2, \tag{18}$$

In view of (16), (17), (18) and condition (3) we derive

$$\|b_n - c\|^2 \le \|d_n - c\|^2 + \gamma(M\gamma - 1)\|(S-I)(Ad_n)\|^2$$
$$\le \|d_n - c\|^2. \tag{19}$$

It follows (15) and (19),we get

$$\|b_n - c\|^2 \le \|d_n - c\|^2 \le \|e_n - c\|^2 \tag{20}$$

Step 4. We prove $\{e_n\}$ is bounded.
Since

$$\|e_{n+1} - c\| = \|g_n u + l_n e_n + y_n b_n - c\|$$
$$\le g_n \|u - c\| + l_n \|e_n - c\| + y_n \|b_n - c\|$$
$$\le g_n \|u - c\| + l_n \|e_n - c\| + y_n \|e_n - c\|$$
$$= g_n \|u - c\| + (1 - g_n)\|e_n - c\|$$
$$\le \max\left\{\|u - c\|, \|e_n - c\|\right\}$$
$$\vdots$$
$$\le \max\left\{\|u - c\|, \|e_0 - c\|\right\} \tag{21}$$

So, $\{e_n\}$ is bounded. Further $\{b_n\}$ and $\{t(d_n)\}$ are also bounded.
Step 5. We show that $\|b_n - b_{n-1}\| \le \|e_n - e_{n-1}\|$.

$$\|b_n - b_{n-1}\|^2 \le \|d_n + \gamma A^*(S-I)(Ad_n) - (d_{n-1} + \gamma A^*(S-I)(Ad_{n-1}))\|^2$$
$$= \|d_n - d_{n-1}\|^2 + \gamma^2 \|A^*(S-I)(Ad_n) - A^*(S-I)(Ad_{n-1})\|^2$$
$$+ 2\gamma \left\langle d_n - d_{n-1}, A^*(S-I)(Ad_n) - A^*(S-I)(Ad_{n-1}) \right\rangle, \tag{22}$$

Since

$$\left\|A^*(S-I)(Ad_n)-A^*(S-I)(Ad_{n-1})\right\|^2$$
$$=\left\langle (S-I)(Ad_n)-(S-I)(Ad_{n-1}), AA^*[(S-I)(Ad_n)-(S-I)(Ad_{n-1})]\right\rangle$$
$$\leq M\left\|(S-I)(Ad_n)-(S-I)(Ad_{n-1})\right\|^2, \tag{23}$$

and by the Lemma 2.2, we get

$$\left\langle d_n-d_{n-1}, A^*(S-I)(Ad)-A^*(S-I)(Ad_{n-1})\right\rangle$$
$$=\left\langle A(d_n-d_{n-1}), (S-I)(Ad_n)-(S-I)(Ad_{n-1})\right\rangle$$
$$\leq -\frac{1}{2}\left\|(S-I)(Ad_n)-(S-I)(Ad_{n-1})\right\|^2. \tag{24}$$

In view of (22), (23) and (24), we have

$$\|b_n-b_{n-1}\|^2 \leq \|d_n-d_{n-1}\|^2+\gamma(\gamma M-1)\|(S-I)(Ad_n)-(S-I)(Ad_{n-1})\|^2$$
$$\leq \|d_n-d_{n-1}\|^2,$$

so,

$$\|b_n-b_{n-1}\| \leq \|d_n-d_{n-1}\|. \tag{25}$$

Since g is a σ-strongly monotone mapping, we have
$\sigma\|d_n-d_{n-1}\|^2 \leq \langle d_n-d_{n-1}, t(d_n)-t(d_{n-1})\rangle \leq \|d_n-d_{n-1}\|\|t(d_n)-t(d_{n-1})\|.$
Therefore, from (11), (12) and (13), we have

$$\|d_n-d_{n-1}\| \leq \frac{1}{\sigma}\|t(d_n)-t(d_{n-1})\|$$
$$=\frac{1}{\sigma}\|P_Y(t(e_n)-\lambda Ke_n)-P_Y(t(e_{n-1})-\lambda Ke_{n-1})\|$$
$$\leq \frac{1}{\sigma}\|(t(e_n)-\lambda Ke_n)-(t(e_{n-1})-\lambda Ke_{n-1})\|$$
$$\leq \frac{1}{\sigma}\|t(e_n)-t(e_{n-1})\|$$
$$\leq \frac{L}{\sigma}\|e_n-e_{n-1}\|$$
$$\leq \|e_n-e_{n-1}\|. \tag{26}$$

By (25) and (26) we know that

$$\|b_n-b_{n-1}\| \leq \|e_n-e_{n-1}\|. \tag{27}$$

Step 6. We show that $\lim_{n\to\infty}\|e_{n+1}-e_n\|=0,\ \lim_{n\to\infty}\|b_n-e_n\|=0$ and $\lim_{n\to\infty}\|e_n-d_n\|=0.$

Let

$$s_n=\frac{e_{n+1}-l_ne_n}{1-l_n}=\frac{g_nu+y_nb_n}{1-l_n},$$

This means that

$e_{n+1}=(1-l_n)s_n+l_ne_n.$

So, we obtain

$$s_{n+1}-s_n=\frac{g_{n+1}u+y_{n+1}b_{n+1}}{1-l_{n+1}}-\frac{g_nu+y_nb_n}{1-l_n}$$
$$=\left(\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right)u+\frac{y_{n+1}}{1-l_{n+1}}(b_{n+1}-b_n)+\left(\frac{y_{n+1}}{1-l_{n+1}}-\frac{y_n}{1-l_n}\right)b_n$$
$$=\left(\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right)u+\frac{y_{n+1}}{1-l_{n+1}}(b_{n+1}-b_n)+\left(\frac{g_n}{1-l_n}-\frac{g_{n+1}}{1-l_{n+1}}\right)b_n$$

From (27), we have

$$\|s_{n+1}-s_n\|-\|e_{n+1}-e_n\|\le\left|\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right|\|u\|+\left|\frac{y_{n+1}}{1-l_{n+1}}\right|\|b_{n+1}-b_n\|$$
$$+\left|\frac{g_n}{1-l_n}-\frac{g_{n+1}}{1-l_{n+1}}\right|\|b_n\|-\|e_{n+1}-e_n\|$$
$$=\left|\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right|(\|u\|+\|b_n\|)+\left|\frac{y_{n+1}}{1-l_{n+1}}\right|\|b_{n+1}-b_n\|-\|e_{n+1}-e_n\|$$
$$\le\left|\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right|(\|u\|+\|b_n\|)+\left|\frac{y_{n+1}}{1-l_{n+1}}\right|\|e_{n+1}-e_n\|-\|e_{n+1}-e_n\|$$
$$\le\left|\frac{g_{n+1}}{1-l_{n+1}}-\frac{g_n}{1-l_n}\right|(\|u\|+\|b_n\|).$$

Since $\{b_n\}$ is bounded, it can be concluded from condition (ii) that
$\limsup_{n\to\infty}(\|s_{n+1}-s_n\|-\|e_{n+1}-e_n\|)\le 0,$
and by Lemma 2.3 and condition (iv) , we get
$\lim_{n\to\infty}\|s_n-e_n\|=0.$
So

$$\lim_{n\to\infty}\|e_{n+1}-e_n\|=\lim_{n\to\infty}(1-l_n)\|s_n-e_n\|=0. \tag{28}$$

In addition, we know that

$$y_n\|b_n-e_n\|-g_n\|u-e_n\|\le\|g_n(u-e_n)+y_n(b_n-e_n)\|$$
$$=\|g_nu+y_nb_n+l_ne_n-e_n\|$$
$$=\|e_{n+1}-e_n\|,$$

And since $0<\liminf_{n\to\infty}l_n\le\limsup_{n\to\infty}l_n<1$, $g_n+l_n+y_n=1$ and $\lim_{n\to\infty}g_n=0$, it is easy to know that $\lim_{n\to\infty}y_n<1$, so, we also have

$$\lim_{n\to\infty}\|b_n-e_n\|=0. \tag{29}$$

Since

$$\|e_{n+1}-c\|^2=\|g_nu+l_ne_n+y_nb_n-c\|^2$$
$$\le g_n\|u-c\|^2+l_n\|e_n-c\|^2+y_n\|b_n-c\|^2$$
$$\le g_n\|u-c\|^2+l_n\|e_n-c\|^2+y_n[\|d_n-c\|^2+\gamma(M\gamma-1)\|(S-I)(Ad_n)\|^2]$$
$$\le g_n\|u-c\|^2+l_n\|e_n-c\|^2+y_n[\|e_n-c\|^2+\gamma(M\gamma-1)\|(S-I)(Ad_n)\|^2]$$
$$\le g_n\|u-c\|^2+\|e_n-c\|^2+y_n\gamma(M\gamma-1)\|(S-I)(Ad_n)\|^2,$$

Then

$$-y_n\gamma(M\gamma-1)\|(T-I)(Ad_n)\|^2 \le g_n\|u-c\|^2+\|e_n-c\|^2-\|e_{n+1}-c\|^2$$
$$=g_n\|u-c\|^2+(\|e_n-c\|-\|e_{n+1}-c\|)(\|e_n-c\|+\|e_{n+1}-c\|)$$
$$=g_n\|u-c\|^2+\|e_n-e_{n+1}\|(\|e_n-c\|+\|e_{n+1}-c\|)\to 0 \quad (n\to\infty),$$

And

$$\lim_{n\to\infty}\|(S-I)(Ad_n)\|^2=0, \tag{30}$$

And

$$\lim_{n\to\infty}\|b_n-d_n\|\le \lim_{n\to\infty}\|(S-I)(Ad_n)\|^2=0 \tag{31}$$

From (29) and (31) we know that

$$\lim_{n\to\infty}\|e_n-d_n\|=0. \tag{32}$$

Step 7. We prove that $\limsup_{n\to\infty}\langle u-\hat{k}, e_n-\hat{k}\rangle \le 0,$ where $\hat{k}=P_\Omega u.$

Since $\{d_n\}$ is bounded, we have a subsequence $\{d_{n_i}\}$ of $\{d_n\}$ with $d_{n_i} \rightharpoonup d^*$. we hve $\{d_n\} \rightharpoonup d^*$. by Opial property, then, $Ad_n \rightharpoonup Ad^*$. Hence, $Ad^*=SAd^*$ by (30) and S is demi-closed at zero, i.e. $Ad^* \in MEP(G,\varphi)$. From (29) and (32), we also have $\{b_n\} \rightharpoonup d^*$ and $\{e_n\} \rightharpoonup d^*$. Next, we prove that $d^* \in GVI(K,t,Y)$.

Set

$$T(\theta)=\begin{cases} K\theta+N_Y(\theta) & \theta\in Y \\ \varnothing & \theta\notin Y \end{cases}$$

Where $N_Y(\theta)$ is the normal cone of Y at θ. By [25] we get T is maximal t-monotone. Suppose $(\theta,\mu)\in Graph(T)$, by $\mu-K\theta\in N_Y(\theta)$ and $e_n\in Y,$ we get $\langle t(\theta)-t(e_n), \mu-K\theta\rangle \ge 0,$ then,

$$\langle t(\theta)-t(e_n), \lambda\mu-\lambda K\theta\rangle \ge 0. \tag{33}$$

It follows

$t(d_n)=P_Y(t(e_n)-\lambda Ke_n),$

We know

$$\langle t(\theta)-t(d_n), t(d_n)-t(e_n)+\lambda K(e_n)\rangle \ge 0. \tag{34}$$

By (33), (34), we get

$$\begin{aligned}\left\langle t(\theta)-t(e_{n_i}),\lambda\mu\right\rangle &\geq \left\langle t(\theta)-t(e_{n_i}),\lambda Kv\right\rangle \\ &\geq \left\langle t(\theta)-t(e_{n_i}),\lambda Kv\right\rangle - \left\langle t(\theta)-t(d_{n_i}),t(d_{n_i})-t(e_{n_i})+\lambda K(e_{n_i})\right\rangle \\ &= \left\langle t(\theta)-t(e_{n_i}),\lambda Kv\right\rangle - \left\langle t(\theta)-t(d_{n_i}),t(d_{n_i})-t(e_{n_i})\right\rangle \\ &\quad - \left\langle t(\theta)-t(d_{n_i}),\lambda K(e_{n_i})\right\rangle \\ &= \left\langle t(\theta)-t(e_{n_i}),\lambda K\theta-\lambda K(e_{n_i})\right\rangle + \left\langle t(\theta)-t(e_{n_i}),\lambda K(e_{n_i})\right\rangle \\ &\quad - \left\langle t(\theta)-t(d_{n_i}),t(d_{n_i})-t(e_{n_i})\right\rangle - \left\langle t(\theta)-t(d_{n_i}),\lambda K(e_{n_i})\right\rangle \\ &\geq -\left\langle t(\theta)-t(d_{n_i}),t(d_{n_i})-t(e_{n_i})\right\rangle + \left\langle t(d_{n_i})-t(e_{n_i}),\lambda K(e_{n_i})\right\rangle \\ &= -\left\langle t(d_{n_i})-t(e_{n_i}),t(\theta)-t(d_{n_i})+\lambda K(e_{n_i})\right\rangle \\ &\geq -\left\|t(d_{n_i})-t(e_{n_i})\right\|\left\|t(\theta)-t(d_{n_i})+\lambda K(e_{n_i})\right\| \\ &\geq -L\left\|d_{n_i}-e_{n_i}\right\|\left\|t(\theta)-t(d_{n_i})+\lambda K(e_{n_i})\right\| \end{aligned} \tag{35}$$

since $e_{n_i} \rightharpoonup d^*$, we have $t(e_{n_i}) \rightharpoonup t(d^*)$, and from (32), we infer that $\left\langle t(\theta)-t(d^*),\lambda\mu\right\rangle \geq 0,$ let $i\to\infty$, so, $\left\langle t(\theta)-t(d^*),\mu-0\right\rangle \geq 0.$ because T is the maximal t-monotonicity, so $d^*\in T^{-1}0$, and $d^*\in GVI(K,t,Y)$. Hence, $d^*\in\Omega$.

So,we obtain

$$\limsup_{n\to\infty}\left\langle u-\hat{k},e_n-\hat{k}\right\rangle = \lim_{i\to\infty}\left\langle u-\hat{k},e_{n_i}-\hat{k}\right\rangle = \left\langle u-\hat{k},d^*-\hat{k}\right\rangle \leq 0, \tag{36}$$

Step 8. Now, we prove $\{e_n\}$ converges strongly to $\hat{k}$.

$$\begin{aligned}\left\|e_{n+1}-\hat{k}\right\|^2 &= \left\langle g_n u + l_n e_n + y_n b_n - \hat{k}, e_{n+1}-\hat{k}\right\rangle \\ &= g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle + l_n\left\langle e_n-\hat{k},e_{n+1}-\hat{k}\right\rangle + y_n\left\langle b_n-\hat{k},e_{n+1}-\hat{k}\right\rangle \\ &\leq g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle + l_n\left\|e_n-\hat{k}\right\|\left\|e_{n+1}-\hat{k}\right\| + y_n\left\|b_n-\hat{k}\right\|\left\|e_{n+1}-\hat{k}\right\| \\ &\leq g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle + \frac{l_n}{2}(\left\|e_n-\hat{k}\right\|^2+\left\|e_{n+1}-\hat{k}\right\|^2) + \frac{y_n}{2}(\left\|b_n-\hat{k}\right\|^2+\left\|e_{n+1}-\hat{k}\right\|^2) \\ &\leq g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle + \frac{1-l_n}{2}(\left\|e_n-\hat{k}\right\|^2+\left\|e_{n+1}-\hat{k}\right\|^2) \\ &\leq g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle + \frac{1-g_n}{2}\left\|e_n-\hat{k}\right\|^2 + \frac{1}{2}\left\|e_{n+1}-\hat{k}\right\|^2,\end{aligned}$$

So

$$\left\|e_{n+1}-\hat{k}\right\|^2 \leq (1-g_n)\left\|e_n-\hat{k}\right\|^2 + 2g_n\left\langle u-\hat{k},e_{n+1}-\hat{k}\right\rangle, \tag{37}$$

It follow condition (ii), (iii), (36) and Lemma 2.4 we have $e_n\to\hat{k},\ d_n$ and b_n are strongly converge $\hat{k}$.

4. Applications

4.1 Application to A Convex Minimization Problem

It's common knowledge that mixed equilibrium problem (3) is simplified to the convex minimization problem (CMP) when $G=0$. Hence, if $\mathcal{W}_1=\mathcal{W}_2,\ Y=D, A=I, t=I, G=0, K=0,$ we can use Theorem 3.1 to solve convex minimization problem: find $c\in Y$ satisfies $\varphi(d)\geq\varphi(c), \forall d\in Y,$ and the listed below result can be directly deduced byTheorem 3.1.

Theorem 4.1. Suppose Y be aclosde convex and non-empty subset of Hilbert space $\mathcal{W}_1$, $S:\mathcal{W}_1$ $\to Y$ is defined as $S(e)=\left\{b\in C:\varphi(d)+\frac{1}{r}\langle d-b,b-e\rangle\geq\varphi(b),\ \forall d\in Y\right\}$, P_Y denotes metric projection of $\mathcal{W}_1$ onto Y, $\gamma\in(0,1).$ Let $\{e_n\}$, $\{b_n\}$ and $\{d_n\}$ be the sequences defined by

$$\begin{cases} e_1\in Y, u\in Y, \\ t(d_n)=P_Y e_n \\ b_n=P_Y((1-\gamma)d_n+\gamma S d_n), \\ e_{n+1}=g_n u+l_n e_n+y_n b_n. \end{cases}$$

where $\{g_n\}$, $\{l_n\}$, $\{y_n\}$ satisfy the listed below conditions:

(a) $g_n+l_n+y_n=1;$

(b) $\lim_{n\to\infty} g_n=0;$

(c) $\sum_{n=0}^{\infty} g_n=\infty$;

(d) $0<\liminf_{n\to\infty} l_n\leq\limsup_{n\to\infty} l_n<1$.

If $\Omega\neq\varnothing$, where $\Omega=\{p:p\in CMP(\varphi)\},$ then $\{e_n\}$ converges strongly to $\hat{k}=P_\Omega u.$

4.2 Application to Split Variational Inequalilty

As is know to all that give a mapping $F:Y\to Y$, let $G(e,d)=\langle Fe,d-e\rangle$ for all $e,d\in Y$. Then $c\in EP(G)$ iff $c\in Y$ is a solution of the variational inequality $\langle Fe,d-e\rangle\geq 0$ for all $d\in Y.$

If $t=I, G(c,d)=\langle Fc,d-c\rangle, \varphi=0$, then, split generalized variational inequality and mixed equilibrium problem proposed by us in this paper reduces to split variational inequality, i.e. find a point $c\in Y$ such that

$$c\in VI(K,Y) \quad and \quad Ac\in VI(F,D).$$

Where $K:Y\to Y, F:D\to D$ be two nonlinear mappings; $A:\mathcal{W}_1\to\mathcal{W}_2$ is a linear and bounded operator.

So, we can use Theorem 3.1 to solve split variation inequality problem and the listed below result can be acquired directly from Theorem 3.1.

Theorem 4.2. Assume that Y and D be two closed convex and non-empty subsets of real Hilbert spaces $\mathcal{W}_1$ and $\mathcal{W}_2$, severally, $K:Y\to Y$ is an ϖ-inverse strongly monotone mapping, $F:D\to D$ is a ρ-inverse strongly monotone mapping, $A:\mathcal{W}_1\to\mathcal{W}_2$ is a linear and bounded operator, the adjoint operator of A is A^*, $\lambda\in(0,2\varpi), \sigma>L, \gamma\in(0,\frac{1}{M})$, where M represent the spectral radius of the operator AA^*, metric projection of $\mathcal{W}_1$ onto Y is P_Y. Let $\{e_n\}$, $\{b_n\}$ and $\{d_n\}$ be the sequences defined by

$$\begin{cases} e_1 \in Y, u \in Y, \\ d_n = P_Y((I-\lambda K)e_n), \\ b_n = P_Y(d_n + \gamma A^*(P_D(I-\lambda F)-I)Ad_n), \\ e_{n+1} = g_n u + l_n e_n + y_n b_n. \end{cases}$$

where $\{g_n\}$, $\{l_n\}$, $\{y_n\}$ satisfy the listed below conditions:

(a) $g_n + l_n + y_n = 1;$

(b) $\lim_{n\to\infty} g_n = 0;$

(c) $\sum_{n=0}^{\infty} g_n = \infty$;

(d) $0 < \liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n < 1$.

If $\Omega \neq \varnothing$, where $\Omega = \{p : p \in VI(K,Y), Ap \in VI(F,D)\}$, then $\{e_n\}$ converges strongly to $\hat{k} = P_\Omega u.$

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References

[1] Stampacchia, G. (1964) Formes bilinearires coercitives sur les ensembles convexes. C.R. Acad. Sci. Paris 258, 4413-4416.

[2] Fichera, G. Problemi elastostatici con vincoli unilaterali, I1 problema di Signorini ambigue condizione al contorno. Attem. Acad. Naz. Lincei. Mem. Cl. Sci. Nat. Sez. Ia 7(8), 91-140.

[3] Bensoussan, A., Lions, J. L. (1982.) Applications of variational inequalities to stochastic control. North-Holland, Amsterdam.

[4] Bensoussan, A. , Lions, J. L. (1984) Impulse control and quasivariational inequalities, Gauthiers Villers, Paris.

[5] Baiocchi, C., Capelo, A. (1984) Variational and quasi-variational inequlities. Wiley, New York.

[6] Crank, J. (1984) Free and Moving Boundary problems. Clarendon Press, Oxford.

[7] Glowinski, R. (1984) Numerical methods for nonlinear variational problems. Springer, Berlin.

[8] Kikuchi, N., Oden, J. T. (1998) Contact problems in elasticity. SIAM, Philadelphia.

[9] Blum, E., Oettli, W. (1994) From optimization and variational inequalities to equilibrium problems. Math. Stud. 63, 123–145.

[10] Noor, M., Oettli, W. (1994) On general nonlinear complementarity problems and quasi equilibrium. Mathematiche (Catanin), 49, 313–346.

[11] Censor, Y. and Elfving, T. (1994) Amultiprojection algorithm using Bregman projections in a product space. Numer, Algorithms, 8, 221-239.

[12] Censor, Y., Bortfels, T. and Trofimov, A. (2006) A unified approach for inversion problem in intensity-modulated radiation therapy. Phys. Med. Biol, 51, 2353-2365.

[13] Censor, Y., Elfving, T., Kopf, N. and Bortfeld, T. (2005) The multiple-sets split feasiblility problem and its applications. Inverse Probl, 21, 2071-2084.

[14] Censor, Y., Motova, A. and Segal, A. (2007) Perturbed projections ans subgradient projiections

for the multiple-sets split feasibility problem. J. Math. Anal. Appl, 327,12441256

[15] Censor, Y., Gibali, A. and Reich, S. (2012) Algorithms for the split variational inequality problem. Numerical Algorithms, vol.59, no.2, pp. 301-323.

[16] Kazmi, K. R. and Rizvi, S. H. (2013) Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. Journal of the Egyptian Mathematical Society , vol.21, pp. 44-51.

[17] Moudafi, A. (2011) A note on the split common fixed-point problem for quasinonexpansive operators. Nonlinear Anal, 74, 4083–4087.

[18] Zhang, X. F., Wang, L., Ma, Z. L. and Qin, L. J. (2015) The strong convergence theorems for split common fixed point problem of asymptotically nonexpansive mappings in Hilbert spaces. Journal of Inequalityes and Applications.

[19] Peng, J. W. and Liou, Y. C. (2009) An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions. Fixed Point theory Appl, Article ID 794178.

[20] Crombez, G. (2006) A hierarchical presentation of operators with fixed points on HIlbert spaces. Numer Funct Anal Optim, 27, 259-277.

[21] Suzuki, T. (2005) Strong convergence theorems for infinite families of nonexpansive mappings in feneral Banach spaces. Fixed Point Theory Appl, 103-123.

[22] Xu, H. K. (2002) Iterative algorithms for nonlinear operators. J. Lond. Math. Soc, 2, 1-17.

[23] Chang, S. S., Kim, J. K. and Wang, X.R. (2010) Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J. Inequal. Appl. V. 14, doi:10.1155/2010/86968.

[24] Geobel, K. and Kirk, W. A. (1990) Topics in metric fixed point theory. Vol. 28 of Cambridge Studies in Advanced Mathematics, Cambrige University Press, Cambridge, UK.

[25] Zhang, U., Chen, J. M. and Hou, Z. B. (2010) Vixcosity approximation methods for nonexpansive mappings and feneralized variational inequlities. Acta Math. Sin, 53, 691-698.