Improvising Differentiation from the View of Integral Calculation

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Abstract: In the method of indefinite integral, the first type of integral integration method and the partial integral method use the improvising differentiation at the same time. At present, the improvising differentiation is biased from the perspective of differential calculus. This paper proposes a new method to learn improvising differentiation from the point of view of integral calculation, in order to help the students overcome learning difficulty.

1. Required Knowledge

To overcome improvising differentiation, you must have the following knowledge points: (1) recognize the composite function; (2) grasp the structural decomposition of the composite function; (3) understand the basic elementary function integral formula; (4) understanding the relationship between differentiability and derivation of functions.

1.1 Recognize the Composite Function

Definition 1 Let the function $y = f(u)$ have a domain of D_f, the function $u = g(x)$ has a domain D_g, and the value field is R_g and $R_g \subset D_f$. The function determined by the following formula:

$$y = f[g(x)], \qquad x \in D_g$$

It is called a composite function composed of function $u = g(x)$ and function $y = f(u)$, and variable u is called intermediate variable [1]. For convenience of description, $y = f(u)$ is called the outer function of the composite function, and $u = g(x)$ is called the inner function of the composite function.

It can be seen from the above definition that under the condition of satisfying $R_g \subset D_f$, one function independent variable is replaced by another function expression, and the new function is a composite function. In general, the outer function $y = f(u)$ is a basic elementary function [1], and the inner function $u = g(x)$ is a basic elementary function or a compound function or a basic operation form of a basic elementary function.

Example 1 Determine whether function A is a composite function?

Solution: It can be known from the analytic formula of the function that the function u is the independent function $\cos x$ of the power function $y = e^u$, so this function is a compound function.

1.2 The structural decomposition of the composite function

The decomposition of a compound function is to give a compound function and find out which functions are compounded by the compound function. In general, it needs to be decomposed layer by layer from the outermost layer to the innermost layer, and it is guaranteed that each layer function decomposed is a basic elementary function or a four-order operation form of a basic elementary function.

Example 2 Decomposing the composite function $y = (2x+1)^8$

Solution: The outermost layer is a form of power function, which is decomposed into $y=u^8$, then $u=2x+1$, which decomposes two layers of functions. One layer is the basic elementary function $y=u^8$, and the second layer is the four elementary form of the basic elementary function $u=2x+1$. Therefore, $y=(2x+1)^8$ is $y=u^8$ Combined with $u=2x+1$.

1.3 Understand the basic elementary function integral formula.

There are 13 basic integral formulas. In the 188 pages of the literature [1], the independent variable x in the basic formula is regarded as any variable form, that is, it can be regarded as such a form: $\int \cos\square d\square = \sin\square + C$, $\int \frac{1}{\square} d\square = \ln|\square| + C$,….

As long as the expression in □ is the same, you can get the conclusion of the formula, otherwise the formula does not hold.

Example 3 Seeking $\int \cos 2x \ d(2x)$

Solution: Available from the basic integral formula, $\int \cos 2x \ d2x = \sin 2x + C$

Example 4 Seeking $\int \cos 2x \ dx$

Solution: $\int \cos 2x \ dx = \frac{1}{2}\int \cos 2x \ d2x = \frac{1}{2}\sin 2x + C$

1.4 Understanding the relationship between differentiability and derivation of functions.

A necessary and sufficient condition for the function $f(x)$ to be different at point x is that the function $f(x)$ is derivable at point x and the subdivision of the function $f(x)$ at point x is:

$$dy = df(x) = f'(x)dx \tag{1}$$

Through this necessary and sufficient condition, you can get $f'(x)dx = df(x)$, which is a function multiplied by the differentiation of the independent variable, which may be the differentiation of a certain function.

Example 5 $\cos x dx = d(\ )$

Solution: From the divisible and derivable relationship, the derivative formula shows that, $\cos x dx = d\sin x$

2. Improvising differentiation

The improvising differentiation is mainly embodied in the integration by substitution and the integration by parts in the calculation of indefinite integral. The following is an example of integration by substitution.

2.1 Integration by substitution:

Theorem 1

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx \overset{improvising\ differentiation}{=} \int f[\varphi(x)]d\varphi(x) \overset{\varphi(x)=u\ substitution}{=} \int f(u)du$$

$$\overset{evaluating\ integral}{=} F(u)+C \overset{u=\varphi(x)\ back\ substitution}{=} F[\varphi(x)]+C \qquad [2]$$

From this theorem, the problem of using improvising differentiation to solve the indefinite integral of a unary function can be done in the following steps:

(1) Judgment: To determine whether you can use improvising differentiation, you need to judge three aspects: ① Observe whether there is a composite function $f[\varphi(x)]$ in the given integrand $g(x)$. ② Observe whether the remaining part of $g(x)$ can be made into the differential $d\varphi(x)$ of the function $f[\varphi(x)]$ in the composite function except for the composite function, that is,

whether there is a constant multiple relationship between the remaining part $g(x)$ and the differential $d\varphi(x)$ of the function $f[\varphi(x)]$ in the composite function. ③ After considering $\varphi(x)$ as a new variable, you can find the result of the integration. If the integrand satisfies the above three conditions at the same time, you can use improvising differentiation.

(2) Improvising differentiation：The remaining part $\varphi'(x)dx$ is made into the differential $d\varphi(x)$ of the function $\varphi(x)$ in the composite function, that is,

$$\int f[\varphi(x)]\varphi'(x)dx \overset{improvising\ differentiation}{=} \int f[\varphi(x)]d\varphi(x). \qquad (2)$$

(3) Substitution：Think of the compound function $\varphi(x)$ in the composite function as a new variable u, and integrate the integral variable x into the definite integral of the new element u, that is, $\int f[\varphi(x)]d\varphi(x) \overset{\varphi(x)=u\ substitution}{=} \int f(u)du$.

(4) Evaluating integral: Calculate the new integral with the new variable u as the integral variable, that is, $\int f(u)du \overset{evaluating\ integral}{=} F(u)+C$.

(5) Back substitution: Change the new variable u back to the expression $\varphi(x)$ of the original variable, and find the original integral result, that is, $F(u)+C \overset{u=\varphi(x)\ back\ substitution}{=} F[\varphi(x)]+C$.

Through the analysis of the above steps, many of the students' confusions have been solved:

(1) Why use improvising differentiation? Because the integrand does not satisfy the basic integral formula.

(2) What kind of form can the integrand satisfy? The simple description is: the integrand has a compound function, and the remaining functions except the compound function can differentiate the function in the compound function, and the function under the new variable has the original function.

(3) What are you going to improvising? The differential form of a function within a compound function.

(4) How to be improvising? The key is that in addition to the compound function outer function, the remaining function is assigned a coefficient, which is divided into the differential of the inner function, which is simply described as: matching coefficient.

Example 6 Calculating $\int (2x+1)^3 dx$

Solution: Follow the steps:

(1) Judgment: There is a compound function $(2x+1)^3$ in the integrand; there is a constant multiple relationship between the remaining part dx and the compound function $2x+1$ differential $d(2x+1)=2dx$. After the improvising differentiation, the internal function is regarded as a new variable and becomes the form of the basic formula of the power function. The integral value can be obtained, so the differential can be used.

(2) Improvising differentiation: Because the purpose is to make the improvising differentiation of the function in the composite function, and know the differential $d(2x+1)=2dx$ of the inner function, so to get the $d(2x+1)=2dx$ out, there is $dx=\frac{1}{2}(2dx)=\frac{1}{2}(2x+1)'dx=\frac{1}{2}d(2x+1)$, then

$$\int (2x+1)^3 dx = \int (2x+1)^3 \frac{1}{2}(2x+1)'dx = \frac{1}{2}\int (2x+1)^3 d(2x+1)$$

(3) Substitution: For $2x+1=u$, the original is converted to $\int (2x+1)^3 dx = \frac{1}{2}\int (2x+1)^3 d(2x+1) = \frac{1}{2}\int u^3 du$.

(4) Evaluating integral: $\int (2x+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{1}{8}u^4 + C$

(5) Back substitution: Returning $u=2x+1$ back, there is $\int(2x+1)^3dx=\frac{1}{2}\int u^3du=\frac{1}{8}u^4+C=\frac{1}{8}(2x+1)^4+C$, the result is obtained.

For example: $\int\frac{\cos\frac{1}{x}}{x^2}dx$, $\int\frac{(\arctan x)^2}{1+x^2}dx$, $\int\frac{\sec^2(\sqrt{x})}{\sqrt{x}}dx$ [3] can be calculated using improvising differentiation, you can try it yourself.

3. Improvising Differentiation from the View of Integral Calculation.

Through the analysis of improvising differentiation, the key to the differential is to judge the part of the $\varphi'(x)dx$ other than the composite function to make up the differential $d\varphi(x)$ of the function in the composite function, that is, there must be a relationship between $\varphi'(x)dx$ and $d\varphi(x)$. By analyzing this key point, as long as the function $\varphi'(x)$ has an original function, the original function has a relationship with the inner function $\varphi(x)$. This converts a problem that is considered from a differential point into a problem that is solved by the integral idea. And in general, the integration of the function $\varphi'(x)$ is simpler. So the method is to turn a complex integral problem into a simple integral problem. The following theorem illustrates the specific principles.

Theorem 2 If $\int g(x)dx=h(x)+C$, and there is a real number $a(a\neq 0)$ and b so that $\varphi(x)=ah(x)+b$ is established, then $\int f[\varphi(x)]g(x)dx=\frac{1}{a}\int f[\varphi(x)]d\varphi(x)$ is established.

Prove:

$$\int f[\varphi(x)]g(x)dx=\int f[\varphi(x)]dh(x)=\frac{1}{a}\int f[\varphi(x)]d[\varphi(x)-b]=\frac{1}{a}\int f[\varphi(x)]d\varphi(x).$$

It can be seen from the above theorem that using the integral idea to make improvising differentiation, only the remaining functions other than the composite function in the integrand are needed to be integrated, and the result of the integration is compared with the internal function of the composite function. If there is a linear relationship, you can use improvising differentiation, and the coefficient is the reciprocal $\frac{1}{a}$ of the real a. If there is no linear relationship, you can't use improvising differentiation.

Example 7 $\int(2x+1)^3dx$

Solution: This function is a compound function. Except for the composite function, there is 1 left. It is integrated with $\int 1dx=x+C$, and an original function is obtained as x. x and internal function $2x+1$ are compared to have real number $a=2, b=1$, so that $2x+1=2\cdot x+1$. Therefore, the rest can be made into the form of an inner function, with $\int(2x+1)^3dx=\frac{1}{2}\int(2x+1)^3d(2x+1)$, let $2x+1=u$, that is, $\int(2x+1)^3dx=\frac{1}{2}\int u^3du$, find the integral $\int(2x+1)^3dx=\frac{1}{2}\int u^3du=\frac{1}{8}u^4+C$. Return $u=2x+1$ back, there is $\int(2x+1)^3dx=\frac{1}{2}\int u^3du=\frac{1}{8}u^4+C=\frac{1}{8}(2x+1)^4+C$.

Example 8 $\int\sin^2 x\cos xdx$

Solution: In addition to the composite function $\cos x$, the integral is $\int\cos xdx=\sin x+C$ and the original function is $\sin x$. Compared with the function $\sin x$ in the composite function, there is a constant $a=1, b=0$ to make $\sin x=1\cdot\sin x+0$, so improvising differentiation can be used, and the result is $\int\sin^2 x\cos xdx=\int\sin^2 xd\sin x$.

Through the above analysis, using the integral idea to make improvising differentiation is positive thinking, and using differential thinking to make the improvising differentiation is reverse thinking. Therefore, the integral idea is more easily accepted by students. On the other hand, the integral part of the integral idea is relatively simple, and can be calculated by the formula, and the students with poor foundation can also use it.

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